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Analysis of the stress-strain state of a radially inhomogeneous transversely isotropic sphere with a fixed side surface

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Introduction. The paper considers an axisymmetric problem of elasticity theory for a radially inhomogeneous transversely isotropic nonclosed sphere containing none of the 0 and π poles. It is believed that the elastic moduli are linear functions of the radius of the sphere. It is assumed that the side surface of the sphere is fixed, and arbitrary stresses are given on the conic sections, leaving the sphere in equilibrium. The work objective is an asymptotic analysis of the problem of elasticity theory for a radially inhomogeneous transversely isotropic sphere of small thickness, and a study of a three-dimensional stress-strain state based on this analysis.

Materials and Methods. The three-dimensional stress-strain state is investigated on the basis of the equations of elasticity theory by the method of homogeneous solutions and asymptotic analysis.

Research Results. After the homogeneous boundary conditions set on the side surfaces of the sphere are met, a characteristic equation is obtained, and its roots are classified with respect to a small parameter characterizing the thickness of the sphere. The corresponding asymptotic solutions depending on the roots of the characteristic equation are constructed. It is shown that the solutions corresponding to a countable set of roots have the character of a boundary layer localized in conic slices. The branching of the roots generates new solutions that are characteristic only for a transversely isotropic radially inhomogeneous sphere. A weakly damping boundary layer solution appears, which can penetrate deep away from the conical sections and change the picture of the stress-strain state.

Discussion and Conclusions. Based on the solutions constructed, it is possible to determine the applicability areas of existing applied theories and propose a new more refined applied theory for a radially inhomogeneous transversely isotropic spherical shell.

Keywords: equilibrium equations, Legendre equations, radially inhomogeneous sphere, characteristic equation, boundary layer solutions, variational principle, applied theory, reduction method.

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Introduction. One of the properties of materials affecting the stress-strain state of elastic bodies is their heterogeneity. Investigating the stress-strain state of inhomogeneous bodies based on three-dimensional equations of elasticity theory is associated with significant mathematical difficulties.

A number of studies are devoted to the investigation of three-dimensional problems of elasticity theory for the sphere.

In [1], based on the equations of elasticity theory for the sphere, a general solution satisfying the boundary conditions on the contour in the sense of Saint-Venant was obtained, the stress-strain state of the sphere was analyzed. In [2], based on the equations of elasticity theory for a thick isotropic sphere, homogeneous solutions depending on the roots of the transcendental equation are constructed. In [3], on the basis of solving three-dimensional problems of

elasticity theory for a sphere of small thickness, the accuracy of existing applied theories is studied, and a method for constructing refined applied theories is given. A three-dimensional asymptotic theory of a transversally isotropic spherical shell of small thickness is presented in [4]. An analysis of the three-dimensional stress-strain state of a three-layer sphere with a soft filler is described in [5]. In [6], the torsion problem is studied by the method of homogeneous solutions for a radially inhomogeneous transversally isotropic sphere of small thickness, when the elastic characteristics are changed by linear, quadratic and inversely quadratic laws along the radius. In [7], the torsion problem for a radially layered sphere with an arbitrary number of alternating hard and soft layers is studied. The existence of weakly damping boundary-layer solutions and a possible violation of the Saint-Venant principle in its classical formulation are shown. An applied theory of torsion of a radially layered sphere is constructed, which adequately takes into account the emerging features. In [8], the problem of elasticity theory for a radially inhomogeneous hollow ball is investigated using the finite element method and spline collocation. In [8], using the finite element method and spline collocation, the problem of elasticity theory for a radially inhomogeneous hollow ball is studied. The results obtained by finite element methods and spline collocation are compared. The axisymmetric problem of the theory of elasticity for a radially inhomogeneous transversally isotropic sphere of small thickness is studied by the method of asymptotic integration of the equations of the theory of elasticity in [9]. Inhomogeneous and homogeneous solutions are constructed. The nature of the stress-strain state is established. In [10], an axisymmetric problem of elasticity theory for a sphere of small thickness with variable elasticity moduli is considered by the method of homogeneous solutions. Asymptotic formulas for displacements and stresses are obtained, which provides calculating the three-dimensional stress-strain state of a radially inhomogeneous sphere.

Materials and Methods. Deformation is considered within the framework of the linear theory of elasticity of a nonclosed sphere, whose material is transversely isotropic and inhomogeneous along the radial coordinate. The thickness of the hollow sphere is assumed to be small compared to the radius and size along the arc coordinate. Boundary conditions are considered that provide solving the problem in an axisymmetric formulation. We assume that the sphere does not contain any of the poles 0 and π . In the spherical coordinate system, the area occupied by the sphere will be denoted by $\Gamma = \{r \in [r_1; r_2], \theta \in [\theta_1; \theta_2], \phi \in [0; 2\pi]\}$.

The linear dependence of the elastic properties of the material along the radius is considered:

$$A_{11} = a_{11}^{(0)}r, \quad A_{12} = a_{12}^{(0)}r, \quad A_{22} = a_{22}^{(0)}r, \quad A_{23} = a_{23}^{(0)}r, \quad A_{44} = a_{44}^{(0)}r, \quad (1)$$

where $a_{11}^{(0)}, a_{12}^{(0)}, a_{22}^{(0)}, a_{23}^{(0)}, a_{44}^{(0)}$ — some constants.

The system of equilibrium equations in the absence of body forces in a spherical coordinate system r, θ, ϕ has the form [11]:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{2\sigma_{rr} - \sigma_{\phi\phi} - \sigma_{\theta\theta} + \sigma_{r\theta} \operatorname{ctg} \theta}{r} = 0, \quad (2)$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{3\sigma_{r\theta} + (\sigma_{\theta\theta} - \sigma_{\phi\phi}) \operatorname{ctg} \theta}{r} = 0, \quad (3)$$

where, $\sigma_{rr}, \sigma_{r\theta}, \sigma_{\phi\phi}, \sigma_{\theta\theta}$ — stress tensor components, which are expressed in terms of displacement vector components $v_r = v_r(r, \theta)$, $v_\theta = v_\theta(r, \theta)$ as follows [4]:

$$\sigma_{rr} = A_{11} \frac{\partial v_r}{\partial r} + \frac{A_{12}}{r} \left(v_\theta \operatorname{ctg} \theta + 2v_r + \frac{\partial v_\theta}{\partial \theta} \right), \quad (5)$$

$$\sigma_{\phi\phi} = A_{12} \frac{\partial v_r}{\partial r} + \frac{1}{r} \left[(A_{22} + A_{23})v_r + A_{22}v_\theta \operatorname{ctg} \theta + A_{23} \frac{\partial v_\theta}{\partial \theta} \right], \quad (6)$$

$$\sigma_{\theta\theta} = A_{12} \frac{\partial v_r}{\partial r} + \frac{1}{r} \left[(A_{22} + A_{23})v_r + A_{23}v_\theta \operatorname{ctg} \theta + A_{22} \frac{\partial v_\theta}{\partial \theta} \right], \quad (7)$$

$$\sigma_{r\theta} = A_{44} \left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right). \quad (8)$$

Substituting (5)–(8) into (2)–(3), taking into account (1), we obtain the equations of equilibrium in displacements.

$$\left\{ \begin{aligned} & b_{11}^{(0)} \left(\frac{\partial^2 u_\rho}{\partial \rho^2} + 2\varepsilon \frac{\partial u_\rho}{\partial \rho} \right) + 2 \left(2b_{12}^{(0)} - b_{22}^{(0)} - b_{23}^{(0)} \right) \varepsilon^2 u_\rho + \left(2b_{12}^{(0)} - b_{23}^{(0)} - b_{22}^{(0)} - b_{44}^{(0)} \right) \times \\ & \times \varepsilon^2 \left(\frac{\partial u_\theta}{\partial \theta} + u_\theta \operatorname{ctg} \theta \right) + \varepsilon \left(b_{44}^{(0)} + b_{12}^{(0)} \right) \left(\frac{\partial u_\theta}{\partial \rho} \operatorname{ctg} \theta + \frac{\partial^2 u_\theta}{\partial \rho \partial \theta} \right) + \varepsilon^2 b_{44}^{(0)} \left(\frac{\partial^2 u_\rho}{\partial \theta^2} + \frac{\partial u_\rho}{\partial \theta} \operatorname{ctg} \theta \right) = 0, \\ & b_{44}^{(0)} \left(\frac{\partial^2 u_\theta}{\partial \rho^2} + 2\varepsilon \frac{\partial u_\theta}{\partial \rho} - 3\varepsilon^2 u_\theta + \varepsilon \frac{\partial^2 u_\rho}{\partial \rho \partial \theta} \right) + \left(3b_{44}^{(0)} + b_{22}^{(0)} + b_{23}^{(0)} \right) \varepsilon^2 \frac{\partial u_\rho}{\partial \theta} + \\ & + \varepsilon^2 b_{22}^{(0)} \left(\frac{\partial u_\theta}{\partial \theta} \operatorname{ctg} \theta + \frac{\partial^2 u_\theta}{\partial \theta^2} \right) + \varepsilon b_{12}^{(0)} \frac{\partial^2 u_\rho}{\partial \rho \partial \theta} - \varepsilon^2 \left(b_{22}^{(0)} \operatorname{ctg}^2 \theta + b_{23}^{(0)} \right) u_\theta = 0. \end{aligned} \right. \quad (9)$$

$$\quad (10)$$

Here, $\rho = \frac{1}{\varepsilon} \ln \left(\frac{r}{r_0} \right)$ — new dimensionless variable; $\varepsilon = \frac{1}{2} \ln \left(\frac{r_2}{r_1} \right)$ — a small parameter characterizing the thickness of the sphere; $r_0 = \sqrt{r_1 r_2}$; $\rho \in [-1; 1]$; $u_\rho = \frac{v_\rho}{r_0}$, $u_\theta = \frac{v_\theta}{r_0}$, $b_{ij}^{(0)} = \frac{a_{ij}^{(0)} r_0}{G_0}$ — dimensionless quantities; G_0 — some parameter having the dimension of the elasticity modulus.

We assume that the lateral part of the sphere boundary is fixed, i.e.,

$$u_\rho|_{\rho=\pm 1} = 0, \quad (11)$$

$$u_\theta|_{\rho=\pm 1} = 0. \quad (12)$$

We assume that at the ends of the sphere (on the conical sections), the stresses are specified

$$\sigma_{\theta\theta}|_{\theta=\theta_s} = f_{1s}(\rho), \quad \sigma_{\rho\theta}|_{\theta=\theta_s} = f_{2s}(\rho). \quad (13)$$

Here, $f_{1s}(\rho)$, $f_{2s}(\rho)$ ($s=1;2$) — sufficiently smooth functions that satisfy the equilibrium conditions.

Solutions (9), (10) are sought in the form [3, 4]:

$$u_\rho(\rho, \theta) = a(\rho)m(\theta); \quad u_\theta(\rho, \theta) = d(\rho)m'(\theta), \quad (14)$$

Where function $m(\theta)$ satisfies the Legendre equation:

$$m''(\theta) + \operatorname{ctg} \theta \cdot m'(\theta) + \left(z^2 - \frac{1}{4} \right) m(\theta) = 0. \quad (15)$$

After substituting (14) into (9), (10), (11), (12), taking into account (15), we obtain:

$$\begin{aligned} & b_{11}^{(0)} a''(\rho) + 2\varepsilon b_{11}^{(0)} a'(\rho) + \varepsilon^2 \left[\left(4b_{12}^{(0)} - 2b_{22}^{(0)} - 2b_{23}^{(0)} \right) - \left(z^2 - \frac{1}{4} \right) b_{44}^{(0)} \right] a(\rho) - \\ & - \left(z^2 - \frac{1}{4} \right) \varepsilon \left[\left(b_{44}^{(0)} + b_{12}^{(0)} \right) d'(\rho) - \varepsilon \left(b_{44}^{(0)} + b_{22}^{(0)} + b_{23}^{(0)} - 2b_{12}^{(0)} \right) d(\rho) \right] = 0, \end{aligned} \quad (16)$$

$$\begin{aligned} & b_{44}^{(0)} \left(d''(\rho) + 2\varepsilon d'(\rho) \right) - \varepsilon^2 \left[\left(z^2 - \frac{1}{4} \right) b_{22}^{(0)} + \left(b_{23}^{(0)} - b_{22}^{(0)} + 3b_{44}^{(0)} \right) \right] d(\rho) + \\ & + \varepsilon^2 \left(3b_{44}^{(0)} + b_{22}^{(0)} + b_{23}^{(0)} \right) a(\rho) + \varepsilon \left(b_{44}^{(0)} + b_{12}^{(0)} \right) a'(\rho) = 0, \end{aligned} \quad (17)$$

$$a(\rho) = 0, \text{ при } \rho = \pm 1. \quad (18)$$

$$d(\rho) = 0, \text{ при } \rho = \pm 1. \quad (19)$$

The solution to system (16), (17) has the form:

$$a(\rho) = e^{-\varepsilon\rho} \left[p_1 e^{\varepsilon s_1 \rho} C_1 + p_1 e^{-\varepsilon s_1 \rho} C_2 + p_2 e^{\varepsilon s_2 \rho} C_3 + p_2 e^{-\varepsilon s_2 \rho} C_4 \right], \quad (20)$$

$$d(\rho) = e^{-\varepsilon\rho} \left[t_1 e^{\varepsilon s_1 \rho} C_1 + q_1 e^{-\varepsilon s_1 \rho} C_2 + t_2 e^{\varepsilon s_2 \rho} C_3 + q_2 e^{-\varepsilon s_2 \rho} C_4 \right], \quad (21)$$

where C_n ($n=1,4$) — arbitrary constants,

$$p_k = b_{44}^{(0)} s_k^2 - \left(z^2 - \frac{1}{4} \right) b_{22}^{(0)} + \left(b_{22}^{(0)} - b_{23}^{(0)} - 4b_{44}^{(0)} \right);$$

$$t_k = - \left(b_{44}^{(0)} + b_{12}^{(0)} \right) s_k - \left(2b_{44}^{(0)} + b_{22}^{(0)} + b_{23}^{(0)} - b_{12}^{(0)} \right);$$

$q_k = (b_{44}^{(0)} + b_{12}^{(0)})s_k - (2b_{44}^{(0)} + b_{22}^{(0)} + b_{23}^{(0)} - b_{12}^{(0)})$; s_k — equation roots

$$\begin{aligned} & b_{11}^{(0)}b_{44}^{(0)}s^4 + \left[\left(z^2 - \frac{1}{4} \right) \left((b_{12}^{(0)})^2 + 2b_{44}^{(0)}b_{12}^{(0)} - b_{11}^{(0)}b_{22}^{(0)} \right) + b_{11}^{(0)}b_{22}^{(0)} - 5b_{11}^{(0)}b_{44}^{(0)} + 4b_{12}^{(0)}b_{44}^{(0)} - \right. \\ & \left. - 2b_{22}^{(0)}b_{44}^{(0)} - b_{11}^{(0)}b_{23}^{(0)} - 2b_{23}^{(0)}b_{44}^{(0)} \right] s^2 + \left[\left(z^2 - \frac{1}{4} \right) b_{44}^{(0)} - 4b_{12}^{(0)} + 2b_{22}^{(0)} + 2b_{23}^{(0)} + b_{11}^{(0)} \right] \times \\ & \times \left[\left(z^2 - \frac{1}{4} \right) b_{22}^{(0)} + b_{23}^{(0)} - b_{22}^{(0)} + 4b_{44}^{(0)} \right] - \left(z^2 - \frac{1}{4} \right) (2b_{44}^{(0)} + b_{22}^{(0)} + b_{23}^{(0)} - b_{12}^{(0)})^2 = 0. \end{aligned} \quad (22)$$

The system of linear algebraic equations with respect to C_1, C_2, C_3, C_4 , is obtained through satisfying the homogeneous boundary conditions (18), (19). The zero equality of the determinant of this system is a condition for the existence of nonzero solutions and leads to a characteristic equation with respect to spectral parameter

$$\begin{aligned} \Delta(z; \varepsilon) &= (p_1q_2 - p_2q_1)(t_1p_2 - t_2p_1)sh^2(\varepsilon(s_1 + s_2)) + \\ &+ (p_1t_2 - p_2t_1)(p_1q_2 - p_2t_1)sh^2(\varepsilon(s_2 - s_1)) = 0. \end{aligned} \quad (23)$$

Equation (23) has countable set of roots z_k . The general solution to the problem is obtained by summing over the roots of equation (23)

$$u_\rho = \sum_{k=1}^{\infty} M_k a_k(\rho) m_k(\theta), \quad (24)$$

$$u_\theta = \sum_{k=1}^{\infty} M_k d_k(\rho) m'_k(\theta), \quad (25)$$

where

$$\begin{aligned} a_k(\rho) &= e^{-\varepsilon\rho} \left[p_1 (e^{\varepsilon s_1 \rho} \omega_1 - e^{-\varepsilon s_1 \rho} \omega_2) + p_2 (e^{\varepsilon s_2 \rho} \omega_3 - e^{-\varepsilon s_2 \rho} \omega_4) \right], \\ d_k(\rho) &= e^{-\varepsilon\rho} \left[t_1 e^{\varepsilon s_1 \rho} \omega_1 - q_1 e^{-\varepsilon s_1 \rho} \omega_2 + t_2 e^{\varepsilon s_2 \rho} \omega_3 - q_2 e^{-\varepsilon s_2 \rho} \omega_4 \right], \\ \omega_1 &= p_2 q_1 (q_2 - t_2) e^{\varepsilon s_1} - t_2 (p_1 q_2 - p_2 q_1) e^{-\varepsilon(s_1 + 2s_2)} + q_2 (p_1 t_2 - p_2 q_1) e^{\varepsilon(2s_2 - s_1)}, \\ \omega_2 &= p_2 t_1 (q_2 - t_2) e^{-\varepsilon s_1} - t_2 (p_1 q_2 - p_2 t_1) e^{\varepsilon(s_1 - 2s_2)} + q_2 (p_1 t_2 - p_2 t_1) e^{\varepsilon(s_1 + 2s_2)}, \\ \omega_3 &= t_1 (p_1 q_2 - p_2 q_1) e^{-\varepsilon(2s_1 + s_2)} - q_1 (p_1 q_2 - p_2 t_1) e^{\varepsilon(2s_1 - s_2)} + p_1 q_2 (q_1 - t_1) e^{\varepsilon s_2}, \\ \omega_4 &= t_1 (p_1 t_2 - p_2 q_1) e^{\varepsilon(s_2 - 2s_1)} - q_1 (p_1 t_2 - p_2 t_1) e^{\varepsilon(2s_1 + s_2)} + p_1 t_2 (q_1 - t_1) e^{-\varepsilon s_2}. \end{aligned}$$

The set of roots of equation (23) at $\varepsilon \rightarrow 0$ consists of countable sets of roots

$$z_k = \frac{\delta_{0k}}{\varepsilon} + O(\varepsilon). \quad (26)$$

For δ_{0k} , we have:

1⁰. At $b_1 > 0$, $b_1^2 - b_2 > 0$:

$$(s_1 - s_2)(b_{44}^{(0)} - b_{11}^{(0)} s_1 s_2) \sin((s_1 + s_2)\delta) \pm (s_1 + s_2)(b_{44}^{(0)} + b_{11}^{(0)} s_1 s_2) \sin((s_1 - s_2)\delta) = 0, \quad (27)$$

where

$$\begin{aligned} s_1 &= \sqrt{b_1 + \sqrt{b_1^2 - b_2}}; \quad s_2 = \sqrt{b_1 - \sqrt{b_1^2 - b_2}}; \\ b_1 &= (2b_{44}^{(0)}b_{11}^{(0)})^{-1} (2b_{44}^{(0)}b_{12}^{(0)} + (b_{12}^{(0)})^2 - b_{11}^{(0)}b_{22}^{(0)}); \quad b_2 = b_{22}^{(0)}(b_{11}^{(0)})^{-1}. \end{aligned}$$

2⁰. At $b_1 > 0$, $b_1^2 - b_2 < 0$:

$$\begin{aligned} & \beta \left[2b_{11}^{(0)}\alpha^2 - (b_{11}^{(0)}(\alpha^2 - \beta^2) - b_{44}^{(0)}) \right] sh(2\delta\alpha) \pm \\ & \pm \alpha \left[2b_{11}^{(0)}\beta^2 + (b_{11}^{(0)}(\alpha^2 - \beta^2) - b_{44}^{(0)}) \right] \sin(2\delta\beta) = 0, \end{aligned} \quad (28)$$

where

$$\begin{aligned} s_1 &= \sqrt{b_1 + \sqrt{b_1^2 - b_2}} = \pm(\alpha + i\beta); \\ s_2 &= \sqrt{b_1 - \sqrt{b_1^2 - b_2}} = \pm(\alpha - i\beta). \end{aligned}$$

3⁰. At $b_1 > 0$, $b_1^2 = b_2$:

$$(b_{11}^{(0)} s^2 - b_{44}^{(0)}) \sin(2s\delta) \pm 2(b_{11}^{(0)} s^2 + b_{44}^{(0)}) \delta s = 0, \quad (29)$$

where $s = \sqrt{b_1}$.

4⁰. At $b_1 < 0$, $b_1^2 - b_2 > 0$:

$$(s_1 - s_2)(b_{44}^{(0)} - b_{11}^{(0)} s_1 s_2) \operatorname{sh}((s_1 + s_2)\delta) \pm (s_1 + s_2)(b_{44}^{(0)} + b_{11}^{(0)} s_1 s_2) \operatorname{sh}((s_1 - s_2)\delta) = 0, \quad (30)$$

where

$$s_1 = \sqrt{|b_1| - \sqrt{b_1^2 - b_2}}; \quad s_2 = \sqrt{|b_1| + \sqrt{b_1^2 - b_2}}.$$

5⁰. At $b_1 < 0$, $b_1^2 - b_2 < 0$:

$$\begin{aligned} & \beta \left[2b_{11}^{(0)} \alpha^2 - (b_{11}^{(0)} (\alpha^2 - \beta^2) - b_{44}^{(0)}) \right] \sin(2\delta\alpha) \pm \\ & \pm \alpha \left[2b_{11}^{(0)} \beta^2 + (b_{11}^{(0)} (\alpha^2 - \beta^2) - b_{44}^{(0)}) \right] \operatorname{sh}(2\delta\beta) = 0, \end{aligned} \quad (31)$$

where

$$s_1 = \sqrt{|b_1| - \sqrt{b_1^2 - b_2}} = \pm(\alpha - i\beta); \quad s_2 = \sqrt{|b_1| + \sqrt{b_1^2 - b_2}} = \pm(\alpha + i\beta).$$

6⁰. At $b_1 < 0$, $b_1^2 = b_2$:

$$(b_{11}^{(0)} s^2 - b_{44}^{(0)}) \operatorname{sh}(2s\delta) \pm 2(b_{11}^{(0)} s^2 + b_{44}^{(0)}) \delta s = 0, \quad (32)$$

where $s = \sqrt{|b_1|}$.

Equation (27)–(32) has a countable set of solutions.

Let us present an asymptotic construction of solutions corresponding to different groups of roots of the characteristic equation (23). Substituting (26) into (24), (25) and expanding the resulting expressions in powers ε , we have:

1⁰.

$$\begin{aligned} \text{a) } u_p(\rho; \theta) = & \sum_{k=1}^{\infty} E_k^{(1)} \delta_k^5 \left\{ (b_{44}^{(0)} + b_{12}^{(0)}) \left[(b_{44}^{(0)} + b_{11}^{(0)} s_2^2) s_1 \cos(\delta_k s_2) \sin(\delta_k s_1 \rho) - \right. \right. \\ & \left. \left. - (b_{44}^{(0)} + b_{11}^{(0)} s_1^2) s_2 \cos(\delta_k s_1) \sin(\delta_k s_2 \rho) \right] + O(\varepsilon) \right\} m_k(\theta), \end{aligned} \quad (33)$$

$$u_\theta(\rho; \theta) = \sum_{k=1}^{\infty} E_k^{(1)} \delta_k^4 \left\{ (b_{44}^{(0)} + b_{11}^{(0)} s_1^2) (b_{44}^{(0)} + b_{11}^{(0)} s_2^2) [\cos(\delta_k s_2) \cos(\delta_k s_1 \rho) - \cos(\delta_k s_1) \cos(\delta_k s_2 \rho)] + O(\varepsilon) \right\} m'_k(\theta), \quad (34)$$

where δ_{0k} are the solutions to equation

$$(s_1 - s_2)(b_{44}^{(0)} - b_{11}^{(0)} s_1 s_2) \sin((s_1 + s_2)\delta) + (s_1 + s_2)(b_{44}^{(0)} + b_{11}^{(0)} s_1 s_2) \sin((s_1 - s_2)\delta) = 0 \quad (35)$$

$$\begin{aligned} \text{б) } u_p(\rho; \theta) = & \sum_{k=1}^{\infty} E_k^{(2)} \delta_k^5 \left\{ (b_{44}^{(0)} + b_{12}^{(0)}) \left[(b_{44}^{(0)} + b_{11}^{(0)} s_2^2) s_1 \sin(\delta_k s_2) \cos(\delta_k s_1 \rho) - \right. \right. \\ & \left. \left. - (b_{44}^{(0)} + b_{11}^{(0)} s_1^2) s_2 \sin(\delta_k s_1) \cos(\delta_k s_2 \rho) \right] + O(\varepsilon) \right\} m_k(\theta), \end{aligned} \quad (36)$$

$$u_\theta(\rho; \theta) = \sum_{k=1}^{\infty} E_k^{(2)} \delta_k^4 \left\{ (b_{44}^{(0)} + b_{11}^{(0)} s_1^2) (b_{44}^{(0)} + b_{11}^{(0)} s_2^2) [\sin(\delta_k s_1) \sin(\delta_k s_2 \rho) - \sin(\delta_k s_2) \sin(\delta_k s_1 \rho)] + O(\varepsilon) \right\} m'_k(\theta), \quad (37)$$

where δ_{0k} are the solutions to equation

$$(s_1 - s_2)(b_{44}^{(0)} - b_{11}^{(0)} s_1 s_2) \sin((s_1 + s_2)\delta) - (s_1 + s_2)(b_{44}^{(0)} + b_{11}^{(0)} s_1 s_2) \sin((s_1 - s_2)\delta) = 0. \quad (38)$$

$$\begin{aligned}
 \text{a) } u_p(\rho; \theta) = & \sum_{k=1}^{\infty} E_k^{(3)} \delta_k^5 \left(b_{44}^{(0)} + b_{12}^{(0)} \right) \left\{ \left[\beta \sin(\delta_k \beta \rho) sh(\delta_k \alpha \rho) - \alpha \cos(\delta_k \beta \rho) ch(\delta_k \alpha \rho) \right] \cdot \right. \\
 & \cdot \left[2\alpha \beta b_{11}^{(0)} \cos(\delta_k \beta) sh(\delta_k \alpha) + \left(b_{11}^{(0)} (\alpha^2 - \beta^2) - b_{44}^{(0)} \right) \sin(\delta_k \beta) ch(\delta_k \alpha) \right] - \\
 & \left. - \left[\alpha \sin(\delta_k \beta \rho) sh(\delta_k \alpha \rho) + \beta \cos(\delta_k \beta \rho) ch(\delta_k \alpha \rho) \right] \cdot \right. \\
 & \left. \cdot \left[2\alpha \beta b_{11}^{(0)} \sin(\delta_k \beta) ch(\delta_k \alpha) - \left(b_{11}^{(0)} (\alpha^2 - \beta^2) - b_{44}^{(0)} \right) \cos(\delta_k \beta) sh(\delta_k \alpha) \right] + O(\varepsilon) \right\} m_k(\theta), \quad (39)
 \end{aligned}$$

$$\begin{aligned}
 u_\theta(\rho; \theta) = & \sum_{k=1}^{\infty} E_k^{(3)} \delta_k^4 \left\{ \left[2\alpha \beta b_{11}^{(0)} \sin(\delta_k \beta \rho) ch(\delta_k \alpha \rho) - \left(b_{11}^{(0)} (\alpha^2 - \beta^2) - b_{44}^{(0)} \right) \times \right. \right. \\
 & \times \cos(\delta_k \beta \rho) sh(\delta_k \alpha \rho) \left. \right] \left[2\alpha \beta b_{11}^{(0)} \cos(\delta_k \beta) sh(\delta_k \alpha) + \left(b_{11}^{(0)} (\alpha^2 - \beta^2) - b_{44}^{(0)} \right) \sin(\delta_k \beta) ch(\delta_k \alpha) \right] - \\
 & - \left[2\alpha \beta b_{11}^{(0)} \cos(\delta_k \beta \rho) sh(\delta_k \alpha \rho) + \left(b_{11}^{(0)} (\alpha^2 - \beta^2) - b_{44}^{(0)} \right) \sin(\delta_k \beta \rho) ch(\delta_k \alpha \rho) \right] \times \\
 & \times \left[2\alpha \beta b_{11}^{(0)} \sin(\delta_k \beta) ch(\delta_k \alpha) - \left(b_{11}^{(0)} (\alpha^2 - \beta^2) - b_{44}^{(0)} \right) \cos(\delta_k \beta) sh(\delta_k \alpha) \right] + O(\varepsilon) \left\} m'_k(\theta), \quad (40)
 \end{aligned}$$

where δ_{0k} are the solutions to equation

$$\beta \left[2b_{11}^{(0)} \alpha^2 - \left(b_{11}^{(0)} (\alpha^2 - \beta^2) - b_{44}^{(0)} \right) \right] sh(2\delta\alpha) + \alpha \left[2b_{11}^{(0)} \beta^2 + \left(b_{11}^{(0)} (\alpha^2 - \beta^2) - b_{44}^{(0)} \right) \right] \sin(2\delta\beta) = 0. \quad (41)$$

$$\begin{aligned}
 \text{б) } u_p(\rho; \theta) = & \sum_{k=1}^{\infty} E_k^{(4)} \delta_k^5 \left(b_{44}^{(0)} + b_{12}^{(0)} \right) \left\{ \left[\beta \sin(\delta_k \beta \rho) ch(\delta_k \alpha \rho) - \alpha \cos(\delta_k \beta \rho) sh(\delta_k \alpha \rho) \right] \cdot \right. \\
 & \cdot \left[2\alpha \beta b_{11}^{(0)} \cos(\delta_k \beta) ch(\delta_k \alpha) + \left(b_{11}^{(0)} (\alpha^2 - \beta^2) - b_{44}^{(0)} \right) \sin(\delta_k \beta) sh(\delta_k \alpha) \right] - \\
 & - \left[\alpha \sin(\delta_k \beta \rho) ch(\delta_k \alpha \rho) + \beta \cos(\delta_k \beta \rho) sh(\delta_k \alpha \rho) \right] \left[2\alpha \beta b_{11}^{(0)} \sin(\delta_k \beta) sh(\delta_k \alpha) - \right. \\
 & \left. - \left(b_{11}^{(0)} (\alpha^2 - \beta^2) - b_{44}^{(0)} \right) \cos(\delta_k \beta) ch(\delta_k \alpha) \right] + O(\varepsilon) \left\} m_k(\theta), \quad (42)
 \end{aligned}$$

$$\begin{aligned}
 u_\theta(\rho; \theta) = & \sum_{k=1}^{\infty} E_k^{(4)} \delta_k^4 \left\{ \left[2\alpha \beta b_{11}^{(0)} \sin(\delta_k \beta \rho) sh(\delta_k \alpha \rho) - \left(b_{11}^{(0)} (\alpha^2 - \beta^2) - b_{44}^{(0)} \right) \times \right. \right. \\
 & \times \cos(\delta_k \beta \rho) ch(\delta_k \alpha \rho) \left. \right] \left[2\alpha \beta b_{11}^{(0)} \cos(\delta_k \beta) ch(\delta_k \alpha) + \left(b_{11}^{(0)} (\alpha^2 - \beta^2) - b_{44}^{(0)} \right) \sin(\delta_k \beta) sh(\delta_k \alpha) \right] - \\
 & - \left[2\alpha \beta b_{11}^{(0)} \cos(\delta_k \beta \rho) ch(\delta_k \alpha \rho) + \left(b_{11}^{(0)} (\alpha^2 - \beta^2) - b_{44}^{(0)} \right) \sin(\delta_k \beta \rho) sh(\delta_k \alpha \rho) \right] \times \\
 & \times \left[2\alpha \beta b_{11}^{(0)} \sin(\delta_k \beta) sh(\delta_k \alpha) - \left(b_{11}^{(0)} (\alpha^2 - \beta^2) - b_{44}^{(0)} \right) \cos(\delta_k \beta) ch(\delta_k \alpha) \right] + O(\varepsilon) \left\} m'_k(\theta), \quad (43)
 \end{aligned}$$

where δ_{0k} are the solutions to equation

$$\beta \left[2b_{11}^{(0)} \alpha^2 - \left(b_{11}^{(0)} (\alpha^2 - \beta^2) - b_{44}^{(0)} \right) \right] sh(2\delta\alpha) - \alpha \left[2b_{11}^{(0)} \beta^2 + \left(b_{11}^{(0)} (\alpha^2 - \beta^2) - b_{44}^{(0)} \right) \right] \sin(2\delta\beta) = 0. \quad (44)$$

3⁰.

$$\text{a) } u_p(\rho; \theta) = \sum_{k=1}^{\infty} E_k^{(5)} \left\{ \left(b_{11}^{(0)} s^2 + b_{44}^{(0)} \right) \left(\cos(\delta_k s) \cos(\delta_k s \rho) + \rho \sin(\delta_k s) \sin(\delta_k s \rho) \right) + \right.$$

$$+\frac{(b_{11}^{(0)}s^2-b_{44}^{(0)})}{s\delta_k}\sin(\delta_k s)\cos(\delta_k s\rho)+O(\varepsilon)\Big\}m_k(\theta), \quad (45)$$

$$u_\theta(\rho;\theta)=\sum_{k=1}^{\infty}E_k^{(5)}\left\{\frac{(b_{11}^{(0)}s^2+b_{44}^{(0)})^2}{(b_{44}^{(0)}s^2+b_{12}^{(0)})\delta_k s}\times\right. \\ \left.\times(\rho\sin(\delta_k s)\cos(\delta_k s\rho)-\cos(\delta_k s)\sin(\delta_k s\rho))+O(\varepsilon)\right\}m'_k(\theta), \quad (46)$$

where δ_{0k} are the solutions to equation

$$(b_{11}^{(0)}s^2-b_{44}^{(0)})\sin(2s\delta)+2(b_{11}^{(0)}s^2+b_{44}^{(0)})\delta s=0 \quad (47)$$

$$6) \quad u_\rho(\rho;\theta)=\sum_{k=1}^{\infty}E_k^{(6)}\left\{\frac{(b_{11}^{(0)}s^2-b_{44}^{(0)})}{\delta_k s}\cos(\delta_k s)\sin(\delta_k s\rho)-(b_{11}^{(0)}s^2+b_{44}^{(0)})\times\right. \\ \left.\times(\sin(\delta_k s)\sin(\delta_k s\rho)+\rho\cos(\delta_k s)\cos(\delta_k s\rho))+O(\varepsilon)\right\}m_k(\theta) \quad (48)$$

$$u_\theta(\rho;\theta)=\sum_{k=1}^{\infty}E_k^{(6)}\left\{\frac{(b_{11}^{(0)}s^2+b_{44}^{(0)})^2}{(b_{44}^{(0)}+b_{12}^{(0)})\delta_k s}\times\right. \\ \left.\times[\rho\cos(\delta_k s)\sin(\delta_k s\rho)-\sin(\delta_k s)\cos(\delta_k s\rho)]+O(\varepsilon)\right\}m'_k(\theta), \quad (49)$$

where δ_{0k} are the solutions to equation

$$(b_{11}^{(0)}s^2-b_{44}^{(0)})\sin(2s\delta)-2(b_{11}^{(0)}s^2+b_{44}^{(0)})\delta s=0 \quad (50)$$

4⁰. In case $b_1 < 0, b_1^2 - b_2 > 0$, asymptotic formulas for displacements are obtained from (33)–(38) through replacing s_1, s_2 with is_1, is_2 .

5⁰. In case $b_1 < 0, b_1^2 - b_2 < 0$, asymptotic formulas for displacements are obtained from (39)–(44) through replacing s_1, s_2 with is_1, is_2 .

6⁰. In case $b_1 < 0, b_1^2 = b_2$, all asymptotic formulas for displacements are obtained from (45)–(50) through replacing s with is .

For roots (26), the main term of the asymptotic solution of equation (15) at $\varepsilon \rightarrow 0$ takes the form [9, 10]:

$$m_k(\theta)=\begin{cases} \frac{1}{\sqrt{\sin\theta}}\frac{1}{\sqrt[4]{-\delta_{0k}^2}}\exp\left[-\varepsilon^{-1}\sqrt{-\delta_{0k}^2}(\theta-\theta_1)\right](1+O(\varepsilon)); \text{ near } \theta=\theta_1, \\ \frac{1}{\sqrt{\sin\theta}}\frac{1}{\sqrt[4]{-\delta_{0k}^2}}\exp\left[\varepsilon^{-1}\sqrt{-\delta_{0k}^2}(\theta-\theta_2)\right](1+O(\varepsilon)); \text{ near } \theta=\theta_2. \end{cases} \quad (51)$$

We represent displacements in the form:

$$u_\rho(\rho,\theta)=\sum_{k=1}^{\infty}E_k a_k(\rho)m_k(\theta), \quad (52)$$

$$u_\theta(\rho,\theta)=\sum_{k=1}^{\infty}E_k d_k(\rho)m'_k(\theta). \quad (53)$$

We represent stresses $\sigma_{\rho\theta}$ and $\sigma_{\theta\theta}$ in the form:

$$\sigma_{\theta\theta}=\sum_{k=1}^{\infty}E_k\left(\sigma_{1k}^{(1)}(\rho)m_k(\theta)+\sigma_{1k}^{(2)}(\rho)m'_k(\theta)\operatorname{ctg}\theta\right), \quad (54)$$

$$\sigma_{\rho\theta} = \sum_{k=1}^{\infty} E_k \sigma_{2k}(\rho) m'_k(\theta), \quad (55)$$

here,

$$\begin{aligned} \sigma_{1k}^{(1)}(\rho) &= \frac{1}{\varepsilon} \left[b_{12}^{(0)} a'_k(\rho) + \varepsilon (b_{22}^{(0)} + b_{23}^{(0)}) a_k(\rho) - \varepsilon b_{22}^{(0)} \left(z_k^2 - \frac{1}{4} \right) d_k(\rho) \right]; \\ \sigma_{1k}^{(2)}(\rho) &= (b_{23}^{(0)} - b_{22}^{(0)}) d_k(\rho); \\ \sigma_{2k}(\rho) &= \frac{b_{44}^{(0)}}{\varepsilon} [d'_k(\rho) + \varepsilon (a_k(\rho) - d_k(\rho))]. \end{aligned}$$

The character of solutions (33)–(50) depends essentially on the type of roots δ_{0k} . The first boundary-layer terms of these solutions correspond to the Saint-Venant edge effect [4]. In the case of imaginary roots δ_{0k} , these boundary layers have weak damping. Thus, the stress-strain state is far enough away from the ends and significantly depends on them. That is, in this case, the transversely isotropic properties of the inhomogeneous material significantly, in comparison to the isotropic material of the sphere, change the pattern of the stress-strain state. At the same time, for real or complex δ_{0k} , the pattern of the stress-strain state of an inhomogeneous sphere for such materials qualitatively coincides, differing in the decay rate of the above-described Saint-Venant boundary layer solutions of an inhomogeneous plate.

From (51), it turns out that when moving away from the conic sections $\theta = \theta_j (j=1, 2)$, solutions (33)–(50) decrease exponentially.

Since the constructed solutions satisfy the equilibrium equation and boundary conditions on the side surface, the Lagrange variational principle takes the following form [4, 11]:

$$\sum_{j=1}^2 \int_{-1}^1 [(\sigma_{\theta\theta} - f_{1j}(\rho)) \delta u_{\theta} + (\sigma_{\rho\theta} - f_{2j}(\rho)) \delta u_{\rho}] \Big|_{\theta=\theta_j} e^{2\varepsilon\rho} d\rho = 0. \quad (56)$$

Substituting (52)–(55) into (56) and taking δE_k as independent variations, we obtain an infinite system of linear algebraic equations

$$\sum_{k=1}^{\infty} E_k q_{jk} = \tau_j, \quad (j=1, 2, \dots), \quad (57)$$

Here,

$$\begin{aligned} q_{jk} &= \int_{-1}^1 \sigma_{1k}^{(1)}(\rho) d_j(\rho) e^{2\varepsilon\rho} d\rho \left(\sum_{s=1}^2 m_k(\theta_s) m'_j(\theta_s) \right) + \int_{-1}^1 \sigma_{1k}^{(2)}(\rho) d_j(\rho) e^{2\varepsilon\rho} d\rho \times \\ &\times \left(\sum_{s=1}^2 m'_k(\theta_s) m'_j(\theta_s) \operatorname{ctg} \theta_s \right) + \int_{-1}^1 \sigma_{2k}(\rho) a_j(\rho) e^{2\varepsilon\rho} d\rho \left(\sum_{s=1}^2 m'_k(\theta_s) m_j(\theta_s) \right), \\ \tau_j &= \sum_{s=1}^2 \left[m'_j(\theta_s) \int_{-1}^1 f_{1s}^*(\rho) d_j(\rho) e^{2\varepsilon\rho} d\rho + m_j(\theta_s) \int_{-1}^1 f_{2s}(\rho) a_j(\rho) e^{2\varepsilon\rho} d\rho \right]. \end{aligned}$$

System (57) is always solvable under physically meaningful conditions imposed on the right side (57). The solvability and convergence of the reduction method for (57) is proved in [12].

Using the smallness of parameter ε , it is possible to construct asymptotic solutions of system (57).

Research Results. The structure of the stress-strain state of a radially inhomogeneous transversally isotropic sphere of small thickness is analyzed under kinematic conditions on the side surface. It is shown that, in the case of fixing the side surface, the character of the solution is determined by the boundary layers. It is found that the asymptotic decomposition of the stress state starts with a solution describing the Saint-Venant edge effect in the theory of transversally isotropic inhomogeneous plates. In the case of transversal isotropy of the radially inhomogeneous material of the sphere, some boundary layer solutions decay very weakly, they can penetrate deep far from the conic sections and change the pattern of the stress-strain state. Asymptotic relations for displacements and stresses are derived, which provide calculating the three-dimensional stress-strain state of a radially inhomogeneous transversally isotropic sphere

of small thickness with any predetermined accuracy. It is shown that the root branching generates a countable set of new solutions for a transversely isotropic radially inhomogeneous sphere.

Discussion and Conclusions. An asymptotic analysis of the stress-strain state of inhomogeneous shells, based on three-dimensional equations of elasticity theory, makes it possible to establish the limits of application of approximate theories. The identified behavior pattern of the solution far from the ends for different boundary conditions on the side surfaces can become the basis for creating refined applied theories for calculating the deformation of a radially inhomogeneous transversally isotropic spherical shell of small thickness. One of the applications of the asymptotic analysis performed can be the calculation of shells with thin coatings, in which, in this case, a radial inhomogeneity arises [13, 14].

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